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The role of multi-representational learning environments to achieve instrumental genesis in mathematics

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ABSTRACT: This article presents results of research on tertiary education students from TEI of Piraeus, to determine terms and conditions that allow the design of didactic interventions in the field of systems of linear equations. The main prerequisites were the imprint of the related learning obstacles and common difficulties, and the design of the interventions. The research was largely qualitative and results were registered under a modified APOS (action process object schema) typology. The design of the intervention was based on Tall's *...three worlds of mathematics* model and Kolb's cycles of experiential learning. Pedagogic use of ICTs was investigated through MATHEMATICA and its multi-representational computable document formats (CDFs). Conceptual change, procedural understanding, and object knowledge were observed. Instrumental genesis was reported in advanced stages along with improved cognitive flexibility. Contribution of multi-representational environments was found important although not altogether without problems.

INTRODUCTION

Linear algebra is a field in mathematics that presents major difficulties for tertiary undergraduates. Specifically, it presents challenges in their achieving of the level of conceptual knowledge and self-regulated learning. Working with linear algebra entails multiple representational systems including graphics, images, symbolic writing, natural language, etc. There is a rich bibliography testifying on that point exactly [1-10].

Many of the relevant difficulties can be traced to an unsuccessful transition of students to abstract thinking in advanced subjects that seem obscure to them. Linear algebra problems often need abstract thinking and advanced mental modelling abilities. Of course, learning gaps from earlier years do play an important role to the difficulties observed. [11-13].

In the process of teaching an introductory lesson on linear systems to students at the Technological Education Institute of Piraeus (TEI-Piraeus), the research presented has been part of an attempt to improve learning outcomes and solve a didactic puzzle on helping students learn the subject matter in a manner that allows for deeper understanding without limiting their ability to attack complex problems. While introducing educational technology in teaching mathematics at all levels of education has merit on its own, it is its enabling abilities that allow for active, creative and critical teaching that was of interest [14-16]. The supposedly beneficial mediation of certain technological tools and artefacts is quite interesting when combined with their proper use as cognitive tools in the process of learning.

It is a widely held view that such attention to the pedagogic relationship between creatively integrated technology and learning in tertiary education is critical if using digital technology in university curricula is to be successful. This is also true for lower educational levels [13][17-19].

The aim of the research here presented is the study of the results and outcomes of an organised didactic intervention that was offered to a group of students at TEI-Piraeus with the intention of supporting them in rising above their difficulties in understanding concepts related to solving problems that take the form of systems of linear equations. Such accommodations are introduced via specific changes in the traditional learning environment that maintain: a) constructivist design of the learning process; and b) introduction of digital technologies in a way coherent with the adopted constructivist approach in learning and teaching.

One major question for this research proved to be whether instrumental genesis of technology can be achieved in the service of student learning in this specific context.

THEORETICAL FRAMEWORK

The basic theoretical concept behind this work is that of Tall's *...three worlds of mathematics* [20]. According to this concept, there are three distinct realms of mathematical thinking and objects, the embodied world of mathematics, the symbolic world of mathematics and the axiomatic world of mathematics. Subjects and the learning process alike permeate the three worlds that are quite different in their structure, handling and most importantly relate to different conveniences that are easier attained by different kinds of thinkers and learners.

Since the direct location of concepts, processes and even problems in the three worlds presents a number of difficulties, a model better tuned for learning attainment was used as an intermediary capturing tool. The APOS model is compatible with Tall's concepts [21], while it evaluates and categorises knowledge, but it also helps learners' progress in a qualitative manner and is consistent with cognitive constructivism. Its four main elements include: a) action oriented or procedural knowledge; b) process oriented knowledge; c) object oriented knowledge - meaning mathematical objects; and d) conceptual or schema knowledge.

Finally, the actual intervention design was based on Kolb's experiential teaching model [22]. Every intervention was designed to include all four stages: concrete experience and motivation; reflective observation and discourse; deduction, conceptualisation and theorising; and active experimentation, action and production of new learning experience. While the four stages and their sequence are set, the departing stage could vary.

MULTIREPRESENTATIONAL ENVIRONMENTS - MATEMATICA SOFTWARE

The concept of representation, as both an individual and social construct, holds a central position in the field of cognitive science. It is also important in educational practice as in essence, when learning building on their previous experiences, people construct a *personal image of the world*, a system of representations that allows them to assimilate and gradually develop new knowledge realising their personal learning and development.

Transitions and transformations between and within representational systems gain importance in constructing and using mathematical concepts. Multiple representations offer different views of a whole, ascribing a pluralistic nature in such concepts [23]. A computer is a formidable symbol editing and presentation tool and its ability to present concurrently multiple representations of a concept, and even to introduce new dynamic ways of representation, made the study of the role of representations in learning even more imperative. Personal computers have been considered to be cognitive tools of some ability to modify human behaviour [24].

The use of multiple and alternative representations raises a need for assistance in developing connections between representations, setting the ability for free action and transition from one to another in the epicentre of the learning process. Some general traits of a multi-representational technological environment that help students develop internalisations of the relationships between the various representations include:

- The ability to display each time the desirable representation;
- The ability to activate or not visible links between representations allowing the learner to contemplate on them and predict;
- The choice of action in each form of representation with real time feedback in the alternative representations [25].

Even if a computer is lacking in representational and expressive power compared with a human, if provided with appropriate software, it still has the ability to enhance the learner's critical thinking, while he/she reflects on their actions during their interaction with the computer [26]. With the ability to go back and forth between various graphic representations and his/her own actions, the user can focus on specific, partial and general attributes of an event, and to create a more complete and coherent image in his/her mind about it [5][12].

INSTRUMENTATION AND INSTRUMENTALISATION OF TECHNOLOGY

The concept of instrumentation or instrumentalisation of artefacts is connected with cognitive ergonomics and is related to the acquisition of skill in use of a technological instrument [27]. It is a concept that has been transferred from music and economics and a number of French educators have applied it in mathematical didactics [1][17][28][29]. It is, basically, a specification of Vygotsky's idea of the mediating role of artefacts that double as learning tools in the case of ICTs in mathematical education [19].

According to this view, an artefact - be it a screwdriver, a guitar or a typewriter - is not by itself a useful tool. Knowledge, skill, experience and sometimes craft or even art are necessary to turn the artefact into a useful instrument in the hands of people. An evolved and complex relationship between the user, the instrument and the work to be done is prerequisite. The process through which an artefact becomes a useful instrument is called instrumental genesis and is in essence a process of cognitive development.

The artefact is instrumentalised via a process of active appropriation in the part of the user that initiates its transformation to a useful instrument or a meaningful tool. The concept of a cognitive schema, originally from cognitive psychology, can also be applied in the case of the specific knowledge of its use as in any case of complex knowledge, so the term of *instrumentation* or *instrumentalisation schemes* is appropriately used here.

These schemes play an important organising role in relation with problem-solving strategies, their use pattern, concepts and theories that form their basis and technical aspects of the use of instruments and tools. According to Rabardel [27], while instrumental genesis takes place, the subject constructs utilisation schemes of the tool in the sense of cognitive schemes that organise activity through the tool in order to achieve a specific target. Rabardel makes the distinction between usage schemes that are oriented towards managing the tool and instrumented action schemes oriented towards task completion and goal attainment. Guin and Trouche [28] state that the process of instrumental genesis goes both ways and make the distinction between instrumentation during which the tool forwards the thinking process of the user improving by its use his/her way of thinking and instrumentalisation of the artefact where the tool is shaped by the user who adapts and transforms it in a way that it is applicable to specific uses [30].

DATA COLLECTION AND METHOD OF RESEARCH

The eventual progress of student learning is by definition estimated in comparison with their performance in the initial stages of didactic intervention. The study of this progress is related to the particular difficulties and the distinct learning process of each student. Due to that fact, the investigation method of student learning is unavoidably that of case studies.

To maintain a rich and complex understanding of the human behaviour entailed, it is also necessary to study it from multiple perspectives [7]. Triangulation is a means for a more complete capture of such multiplicity, in the sense of using multiple techniques of both data acquisition and data organisation related to the study of aspects of this behaviour.

The methods used in this case included mainly interviews and discussions, independent researcher observation and assessment of appropriate data (documentation). A research interview is described as a discussion between two people, with the starting point being the interviewer and with the specific purpose of acquiring relevant information, with the content specified by the purposes of the research, with systematic description, provision or interpretation [31][32].

In the case here, there were two kinds of interview. The *initial interview* was semi-structured. Interviewees described their earlier educational experiences, their knowledge, their views, their conceptions, their interpretations and their interaction in specific subjects. A number of exploratory questions, regarding mathematics in general, and linear algebra specifically, covered a substantial area of concepts in order to establish a baseline cognitive condition for the students.

Constant-perpetual local or *micro-interviews* were the other category. Open interviews of short duration, usually taking place during plenary sessions and while the intervention was *en route*, usually occurred more than once in each session and were accompanying the formation of a new theory, the emergence of a cognitive conflict or a simple difficulty in understanding or handling on the part of the students. They lasted for a few minutes and their end depended on clarification of the subject, the end of the session and in rare cases a consensus to revisit the subject on another occasion.

The interviews took place in the classroom during the research period and when it was feasible to do so without undermining the intervention and time was available. Due to time limitations on the part of the participating students, it was decided that they would take place every time there was a difficulty in flow or understanding. Even though there was the opportunity for personal interactions, they mostly took place in the presence of the group although general participation varied.

With an initial thematic approach, they were driven by the very answers of the student or students involved, which formed the subsequent questions on the part of the interviewer creating a local vortex of questioning before resuming with the intervention or calling it a day. Students would often also participate with questions of their own, contributions or examples. They participated by asking clarifying or of reflecting nature, questions towards the interviewee or the group.

They allowed students to express their images of basic concepts, processes they applied in solving problems, and also interconnections between concepts. As Clement put it, the interviews gave the researchers an opportunity to collect data on cognitive processes on a level of authentic ideas of each involved subject with only their own mediation - exposing secret structures and cognitive processes [31].

Still, this kind of communication was often unbalanced, while the part of the interviewer, as perceived from the interviewee, forms the answers of the latter to some extent [33]. There is always a danger lurking of hasty questioning that might lead to inaccurate or deceptive information. There is also a chance that the interviewers' stance and attitude will take them further away from the actual students' positions and ideas [34].

Keeping that fact in mind, it was deemed necessary to allow the open group interviews to evolve relatively freely in class discussion when possible. That was easy, due to the nature of the educational intervention that shaped the research. It allowed important references of a meta-cognitive nature, a higher degree of interaction among students and a recording of not only subject attitudes, but also variations of the levels of cognitive functions and interactions.

INTERVENTION RESULTS

The intervention constantly revealed a certain pattern, that of a laboriously and incrementally developing of schemas through the reflective evaluation of multiple experiences. These experiences were connected to each other, but not automatically and not without difficulty. When a breakthrough was achieved by one student though, it expanded virally to the rest of the class once presented by the original receiver of the *revelation*. A good example of this process is described below.

In Session 3, the intervention introduced the application of planes solution and Gaussian elimination in 3 x 3 linear systems and combined it with two activities. In the first activity, two equations were given to the students with three unknowns, which were inserted to the application, where they were represented as two plains. Then, students were asked to modify one equation so that the two plains become parallel. Initially Dimitris and Nikos found it difficult to handle the activity. Dimitris focused on modifying fixed terms, but by watching the plain move without rotating, understood his mistake. Stephen introduced an original theory of how this would be possible and they all started experimenting. All experiments were completed successfully and the students were able to demonstrate understanding of the process during the micro-interviews of meta-cognitive interest that followed.

In the second activity in the first two equations a third was added creating a 3 x 3 system. Students were asked to intervene appropriately in the existing system, modifying the equations in order to achieve an intersection of three planes. Students misled by the existence of intersections in twos wondered whether this fact alone could lead to a solution, ignoring the fact - already identified by them in a previous activity - that the first two equations represent parallel planes and that this fact leads inevitably to the characterisation of the system as impossible, regardless of the choice of the third equation.

The students seemed to have overlooked or at least reflected poorly on the previous conclusion that the two planes are parallel and, in essence, they seemed to view this as a new system of which they had no previous knowledge. It seemed, therefore, that any models they have developed regarding the solutions of a system are not clear cut. Mislead perhaps by the original question of *What is the relative position of the third equation?* that leads to the creation of subsystems, they seemed unable to synthesise the partial conclusions into a final characterisation of the system regarding its solvability.

During the course of the intervention a permanent attachment to proportions between actors could be witnessed, kept or broken, depending on the needs for re-orientation of the planes. Although there is such a need in cases of parallel ones, in cases that the planes intersect each other, there is no reason for the use of ratios. In this case, it was observed that when they were asked to record intersecting planes, they used to keep two coefficients equal and differentiated the third to produce an intersection. This is consistent with Bogomonly's observations on the subject of linear dependence/independence, observations that she ascribes to a process-oriented logic [35].

Such processes, with the related cognitive conflicts, triggered some voluntary interaction with the dynamic environment on the part of the students and, thus, they took advantage of the opportunities it gave them; eventually, they started to experiment on personal models of action as in the case of Nick. Here a point, as a singular solution to the 3 x 3, was displayed starting from three mismatched levels, emerged by varying the coefficients of the variables in order to first identify the intersecting lines of two subsystems which, eventually, lead to the emergence of the unique solution (point).

Nick: *If you make three, not parallel, three planes that are identical and simply change a coefficient for each plane. Do I get planes intersecting on a line?* I say yes. *So we can also get to a point. If I turn one of the two, I understand how I can reach a straight line intersection. The point is still giving me some difficulty*.

And then again: Nick: *The logic to follow is where the lines intersect... Basically if I start messing with b (refers to the second coefficient of the equation he is* messing *with) from this line now I should get a point* ... Here it is! ...*but how do I explain it? I said by rotation. I started with three planes mismatched and messed with the coefficient of x on two planes, they will rotate and we will get a line intersection. And if mess with the third plane, basically on one axis either x or z, then the intersection of that plane with the line should be a point.*

The difficulty is not entirely unexpected. One perceives a point as the intersection of two lines, rather than as the intersection of three planes. This is a point the whole interaction just described seems to affirm. Such combinations of individual conclusions are not reached in the early stages of intervention, but at this point, they follow inexorably from the procession of the steps the student has chosen to implement. The flow is tied to the structure of the computing environment (MATHEMATICA) and leads to successive levels of increased understanding, substantially surpassing the original problem. The final approach, although at a procedural level, leads to a clear understanding of the ease with which his fellow students share an initial level right after the initial presentation.

Regardless of the theoretical framework, in the present study, it was repeatedly observed that multiple representations and approaches:

- facilitated the development of concept images of unexpected completeness when the initial situation of the students was taken into account;
- seemed to maintain student interest; and
- different representations became the stepping point for different students.

This third point was particularly served by another structural feature of the intervention, namely revisitation. There was a revisiting of issues, problems and representations during the intervention, consistent both with Kolb's model used in the design, but also with Bruner's notion of spiral learning, which is consistent with the notion of crossing through Tall's three worlds, which it actually enhances.

The very crossing of the three worlds, a structural element of the intervention, appears to be directly responsible for the quality of the final conceptual schemas developed by the students. This was achieved through the revisiting of concepts through multiple, yet discrete approaches that can be traced to: a) observation/aggregation; b) symbolic approaches (either algebraic or tabular); and finally c) standard form, not necessarily in that order.

CONCLUSIONS

Conceptual change, procedural understanding and object knowledge did not appear incrementally or unexpectedly. Instead they emerged again and again, not as results of the availability of multiple representations, but of the learners engaging with them. They were not only observed emerging within the learning process of a single student, but also acting as contagiously spreading in the class through discussions and group reflection.

Instrumental genesis was reported in advanced stages of the didactical intervention along with improved cognitive flexibility among the students. The contribution of the multi-representational environment was found important although not altogether without problems. Overcoming common mistakes and difficulties at the time of tool set introduction seemed to also help.

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